THURSDAY, OCTOBER 4, 1888.

## DETERMINANTS.

*Teoría Elemental de las Determinantes y sus principales aplicaciones al Algebra y la Geometría.* Por Félix Amorétti y Carlos M. Morales. (Buenos Ayres: Imprenta de M. Biedma, 1888.)

"SOME books," says Bacon, "may be read by deputy, and extracts made of them by others;" and, at any rate so far as English readers are concerned, the work now under review belongs to this category. A very considerable portion of it is taken up with translations of selected passages from Muir's "Treatise on the Theory of Determinants," of which the following is a sample:—

"Teorema III.—*Toda determinante centrosimétrica del orden* $2\mathrm{m}^{\acute{e}simo}$ *es igual á la diferencia de los cuadrados de dos sumas de determinantes menores del orden* $\mathrm{m}^{\acute{e}simo}$ *formadas con las* m *primeras filas.*"

"En efecto; el producto de las dos determinantes factores, por ejemplo, D y D′, en el caso del párrafo 37, es igual á

$$\tfrac{1}{2}\{[D + D']^2 - \tfrac{1}{2}[D - D']^2\},$$

y si D y D′ se desarrollan en función de determinantes de elementos monomios (22), las determinantes de una de las expresiones son iguales á las de la otra. Luego queda demostrado el teorema."

The above is almost word for word the same as § 138 of Muir's "Treatise," which we subjoin for the sake of comparing the translation with the original:—

"*A centro-symmetric determinant of the* $2\mathrm{m}^{th}$ *order is expressible as the difference of the squares of two sums of minors of the* $\mathrm{m}^{th}$ *order formed from the first* m *rows.*

"The product of the two factors, D and D′ say, in the first case of § 137 is equal to

$$\{\tfrac{1}{2}(D + D')\}^2 - \{\tfrac{1}{2}(D - D')\}^2,$$

and when D and D′ are expanded (§ 29) in terms of determinants with monomial elements, the determinants in the one expansion are in magnitude the same as those in the other: hence the theorem."

We notice that Muir's formula is incorrectly printed in the translation; but it is only fair to add that such inaccuracies are rarely met with in the volume before us, which is more free from misprints than the first editions of mathematical books usually are. The translators do not appear to have caught the exact meaning of the words "are in magnitude the same as," which they have changed into "son iguales á." Quantities which are the same in magnitude (though differing, it may be, in sign) they call equal, and are consequently forced to translate the words "are equal" by "son iguales y del mismo signo" as they have done elsewhere in more than one place. But a much worse mistranslation (also from Muir) occurs on p. 85, where the single word "es" is used as the equivalent of "contains the term." A worse mistake than this one could not have been committed, even by those who, according to Hudibras, "translate,—

> Though out of languages, in which
> They understand no part of speech."

The above extract is taken from the second of the three distinct portions, or books, into which the "Teoría Elemental" is divided. The first of these books has to do with determinants in general, the second (consisting mainly of translations from Muir) treats of determinants of special form, and the third is reserved for algebraical and geometrical applications. The nomenclature adopted in the second book differs in some particulars from that employed by Muir. Thus our authors do not follow him in substituting "adjugate" for the more euphonious and more familiar adjective "reciprocal," and they agree with Scott and others in calling those determinants "orthosymmetrical" which Muir names "persymmetric." We think that their name "determinante *hemisimétrica*" is a distinct improvement on the old "*zero-axial skew* determinant," but we cannot see any special reason for speaking of determinants in which all the elements in one row are equal to unity as "determinantes multiples," and we do not consider that the fact of the equality of all the elements in the principal diagonal of any skew determinant is of sufficient importance to necessitate the use of the distinctive appellation "*pseudo-simétrica*" to denote such a skew determinant.

The second book contains most of the principal properties of the various kinds of symmetrical determinants, and of Pfaffians, alternants, circulants, and continuants, but not of compound or functional determinants: these are mentioned, but their properties are not investigated. The short chapter devoted to them merely defines compound determinants, Jacobians, Hessians, and Wrouskians, and then concludes abruptly with these words: "Por más interesantes que sean estas formas, la índole de esta obra no permite entrar en el estudio de ellas, para el cual se recomienda especialmente el notable tratado del profesor R. Scott, 'Determinants,' Cambridge, 1880."

Here our remarks on the second book (which finishes with this sentence) would come to a close if we did not wish to correct a mistake into which the authors have fallen as to the origin of the name *continuants*. These they say (see p. 112) "se denominan continuantes, por sugestión del profesor Sylvester." The real facts of the case are these. Prof. Sylvester was the first to discover the forms called *continuants*, to which he gave the name of *cumulants*. It was Muir who suggested the name *continuant* "as an exceedingly suitable and euphonious abbreviation for *continued-fraction determinant*," and as a "short literal translation of the equivalent term *Kettenbruch-Determinante*, which is the received name in Germany" (vide *American Journal of Mathematics*, vol. i. p. 344: letter from Mr. Muir to Prof. Sylvester on the word continuant, September 4, 1878).

Of the third book we have very little to say. It is nice easy reading for young beginners, and teaches them how to solve systems of linear equations, how to perform eliminations by means of Euler's, Bezont's as modified by Cauchy, or Sylvester's dialytic method, and how to calculate the roots common to two equations or the double roots of a single equation. There is a short chapter in which some of the most simple properties of the resultant of two equations are explained. The last chapter in the book is the only geometrical one; its principal contents are determinant expressions for the area of a triangle, a quadrilateral, and a polygon, in terms of the co-ordinates

© 1888 Nature Publishing Group

of their respective vertices, and some simple trigonometrical formulæ. On p. 173, in this chapter, we notice a curious misprint: in each of three successive formulæ (the usual expressions for the sine, cosine, and tangent of half an angle of a triangle in terms of its sides) a capital V takes the place of the sign of the square root.

The opening paragraph of the first book tells us of the origin of determinants, citing as evidence of their invention by Leibnitz his celebrated letter to L'Hopital, dated April 28, 1693. Their re-discovery by Cramer in 1750, and the rule (for the solution of a system of linear equations) which still bears his name, are next mentioned; but authors of a more modern date are summarily dismissed with the following brief notice:—

"Desde el tiempo de Cramer la teoría de las determinantes ha hecho notables progresos debido á los trabajos de Vandermonde, Laplace, Gauss, Cauchy, Jacobi, Sylvester, Muir, Baltzer y otros, no habiendo rama de las matemáticas en que no haya sido aplicada con ventaja."

We are not, however, left entirely in the dark as to the contributions to the theory made by these writers; for some theorems are called by the names of their respective authors, and a large number of others have these names indicated in brackets. For instance, the proposition which concludes the third chapter in the first book is thus enunciated:—

"Descomponer una determinante de orden $n^{\text{ésimo}}$ en una suma de productos formados cada uno de una determinante de orden $p^{\text{ésimo}}$ y de una determinante de orden $[n-p]^{\text{ésimo}}$ { Laplace }."

This is immediately preceded by—

"Teorema de Cauchy.—Si se elige una fila y una columna de una determinante cualquiera, el elemento común de ellas multiplicado por el respectivo complemento algebraico, más la suma de productos obtenidos multiplicando el producto de un elemento de la fila y de la columna por su respectivo complemento algebraico, es equivalente á la determinante dada."

The way in which these two propositions are treated in the present work will serve to exemplify the methods employed by its compilers for imparting knowledge to their readers. The proof of Laplace's theorem given by Scott, in § 5, chap. iii. of his "Determinants," is clearer than any other we are acquainted with; but it depends on some of the properties of alternate numbers. It is true that these properties are of the simplest kind, but then the notion of alternate numbers is a highly abstract one, quite as much so as the idea of a four-dimensional space. In order, therefore, to convey a clear conception of Laplace's theorem to students of average capacity, our authors have turned it into a problem, and, by considering what Prof. Sylvester calls a simple diagrammatic case, have shown how this problem can be solved, thereby bringing the *theorem* within the grasp of those whose minds are as yet unprepared to revel luxuriously in such abstractions as the alternate numbers.

On the other hand, the proof of Cauchy's theorem and the illustrative example appended to it have been reproduced, with only some slight vertal alterations, from § 62 of Muir's "Determinants," where the theorem in question is presented in a form eminently adapted for elementary instruction.

The first book ends with a rule for the division of determinants, which may be briefly stated thus: To divide $|a_{1n}|$ by $|b_{1n}|$, assume the quotient to be $|x_{1n}|$ and equate each element of the determinant formed by multiplying $|x_{1n}|$ and $|b_{1n}|$ to the corresponding element of $|a_{1n}|$.

The values of the elements $x_{11}, x_{21}, \ldots . x_{nn}$, of the assumed quotient $|x_{1n}|$ will then be determined by solving a system of equations of the form

$$b_{1n}\,x_{1n} + b_{2n}\,x_{2n} + \ldots . + b_{nn}\,x_{nn} = a_{nn}.$$

The article containing this rule should be expunged from all future editions of the work. Its practical inutility becomes apparent when we remember that, on solving the system of equations to which it leads, each $x$ is found in the form of the quotient of two determinants; so that we have to perform many divisions instead of one. Those who are practically engaged in the work of mathematical tuition in the University of Buenos Ayres will doubtless be able to suggest other improvements, and if these suggestions are attended to, students in that University will possess in the second edition of the "Teoría Elemental" an introduction to the theory of determinants written in their own language and suited to their requirements.

In some respects we do not desire to see any improvement. The appearance of the book is as attractive as good paper, wide margins, and a bold clear type can make it. The authors have chosen for their motto the appropriate quotation from Sylvester: "For what is the theory of determinants? It is an algebra upon algebra; a calculus which enables us to combine and foretell the results of algebraical operations, in the same way as algebra enables us to dispense with the performance of the special operations of arithmetic." The table of contents is a model of completeness, and gives the enunciations of the theorems in full instead of merely indicating the pages and articles in which they occur. The volume ends with a selected list of treatises on determinants "que pueden servir de texto y que son dignas de especial mención." This will be of use to students who only want to be told what authors they should read, for the names mentioned are few and well chosen; while those whose object is to improve their acquaintance with the bibliography of determinants may fully satisfy their desire by consulting the two papers by Muir in the *Quarterly Journal of Mathematics* (one of them published in 1881, the other in 1886) to which reference is made.

Responding to the invitation—"agradeceríamos las indicaciones que se nos hicieran sobre omisiones ó errores que no hubiéramos advertido"—we call attention to a slight misprint in this reference, in which the word "Quarterly" has been mis-spelt "Quaterly." With the exception of those previously mentioned, no other *erratum* has come under our notice.

## OUR BOOK SHELF.

*The Geological History of Plants.* By Sir J. W. Dawson, C.M.G., LL.D., F.R.S., &c. 8vo, pp. 290. With Illustrations. "International Scientific Series." (London: Kegan Paul, Trench, and Co., 1888.)

THIS book gives, in a connected form, a summary of the development of the vegetable kingdom in geological time.

© 1888 Nature Publishing Group